Section 13.2: Definite integrals and the Fundamental Theorem of Calculus

Recall: An *indefinite integral* is a function (the general antiderivative)

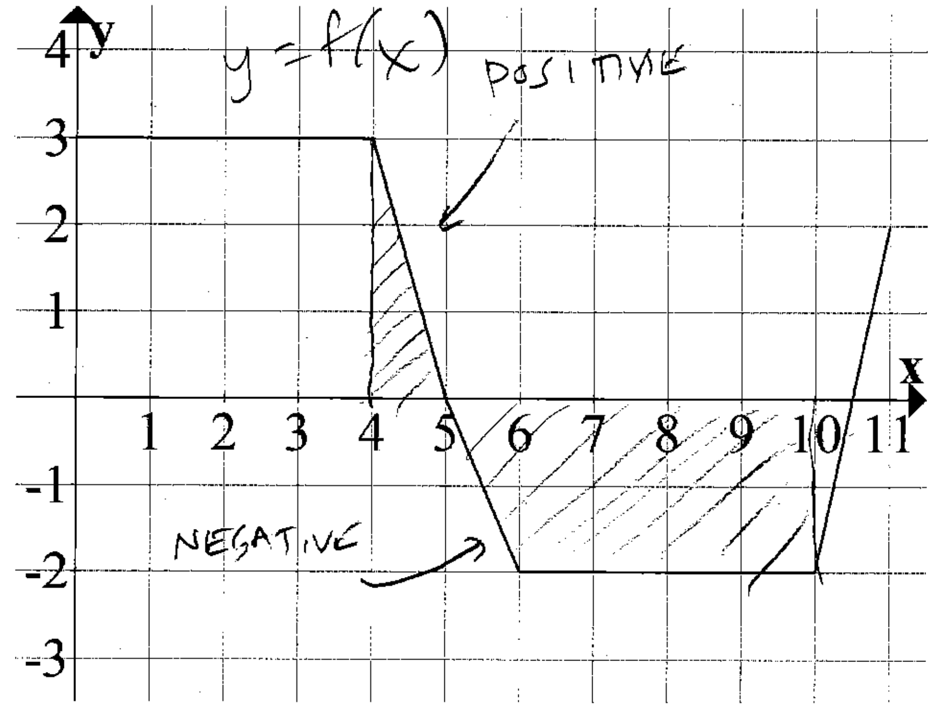
$$\int f(x)dx = F(x) + C$$

New: A *definite integral* is a number that represents *net* area

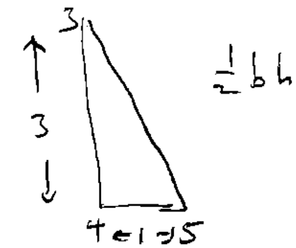
$$\int_a^b f(x)dx = \text{"}net \text{ area between } f(x) \text{ and the } x-\text{axis from } x = a \text{ to } x = b\text{"}$$

<u>*Notes*</u>

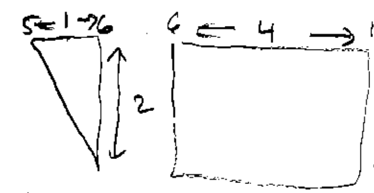
Above the x-axis counts as positive area.
Below the x-axis counts as negative area.
"*a*" and "*b*" are called the *bounds,* or *limits, of integration*.



$$\int_4^5 f(x)dx = \tfrac{1}{2}(1)(3)$$
$$= 3/2 = 1.5$$

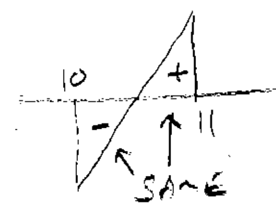


$$\int_5^{10} f(x)dx = -9$$



$\frac{1}{2}(1)(2) + 2 \cdot 4 = 1 + 8 = 9$

$$\int_{10}^{11} f(x)dx = 0$$



Now consider

$$A(m) = \int_0^m f(x)\,dx$$

= "*accumulated* net area from *0* to *m*"

Using the same graph, what is

$$A(0) = \int_0^0 f(x)\,dx = 0$$

increase

$$A(4) = \int_0^4 f(x)\,dx = 3 \cdot 4 = 12$$

increase

$$A(5) = \int_0^5 f(x)\,dx = 12 + 1.5 = 13.5$$

decrease

$$A(8) = \int_0^8 f(x)\,dx = 13.5 - \tfrac{1}{2}(1)(2) - 2\cdot 2$$
$$= 13.5 - 1 - 4 = 8.5$$

Questions/Observations:

Where is *A(m)* increasing/decreasing?

$A(m)$	$\nearrow$		$\searrow$		$\nearrow$
$f(m)$	$+$	5	$-$	10.5	$+$

$A(m)$ increases (adds positive area) for $0 \leq m < 5$ and for $m > 10.5$.

NOTE: $m = 5$ is a local max
$m = 10.5$ is a local min.

See any connections for *A(m)* and *f(x)*?

$A'(m) = f(m)$

$A(m) =$ an antiderivative of $f(m)$

What does $A(5) - A(4)$ represent?

$A(5) - A(4) = \int_4^5 f(x)\,dx = 1.5$

In addition, in the activities you found:

1. "the area under the speed graph"

   equals "the change in distance".

$$\int_a^b s(t)dt = D(b) - D(a)$$

2. "the area under the MR/MC graph" equals "the change in TR/TC"

$$\int_a^b MR(x)dx = TR(b) - TR(a)$$

$$\int_a^b MC(x)dx = TC(b) - TC(a)$$

These are examples of a profound fact about anti-derivatives and areas.

The Fundamental Theorem of Calculus

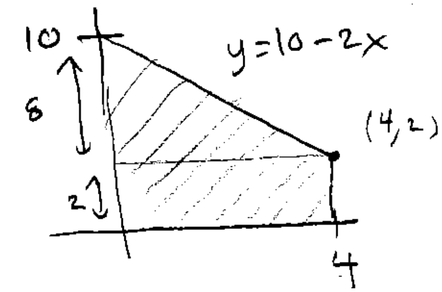
If $F(x)$ is *any* anti-derivative of $f(x)$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

Example: Find the area under

$$MR(x) = 10 - 2x$$

from $x = 0$ to $x = 4$.



2 options

$\boxed{\text{I}}$ GEOMETRY

AREA $= 2\cdot4 + \frac{1}{2}(4)(8) = 8 + 16 = \boxed{24}$

$\boxed{\text{II}}$ USE FTOC.

$$\int_0^4 \underbrace{10 - 2x}\, dx$$

$F(x) = 10x - x^2$

$F(4) = 10(4) - (4)^2 = 24$

$F(0) = 10(0) - (0)^2 = 0$

$$= 10x - x^2 \Big|_0^4$$

$$= \underbrace{\left(10(4) - (4)^2\right)}_{F(4)} - \underbrace{\left(10(0) - (0)^2\right)}_{F(0)}$$

$$= \boxed{24}$$

How to compute definite integrals

Step 1: Find *any* antiderivative, $F(x)$.

(usually we pick C = 0, but you use any C value you want and it doesn't change the answer)

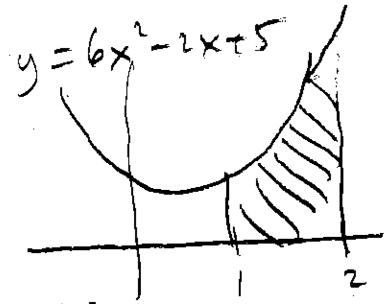
Step 2: Evaluate $F(x)$ at $x = b$ and $x = a$.

Step 3: Subtract

We do all this in one line as follows:

$$\int_a^b f(x)dx = \left. F(x)\right|_a^b = F(b) - F(a)$$

More Examples:



1. $\int_1^2 6x^2 - 2x + 5\, dx$

$$= 6\tfrac{1}{3}x^3 - 2\tfrac{1}{2}x^2 + 5x \Big|_1^2$$

$$= 2x^3 - x^2 + 5x \Big|_1^2$$

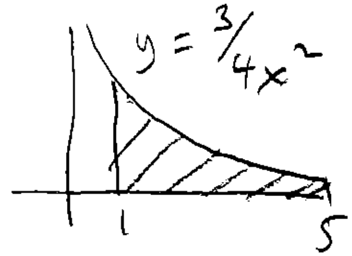
$$= \underbrace{\left(2(2)^3 - (2)^2 + 5(2)\right)}_{F(2)} - \underbrace{\left(2(1)^3 - (1)^2 + 5(1)\right)}_{F(1)}$$

$$= (16 - 4 + 10) - (2 - 1 + 5)$$

$$= 22 - 6$$

$$= \boxed{16}$$

2. $\int_1^5 \frac{3}{4x^2}\,dx \longleftrightarrow$



$= \int_1^5 \frac{3}{4}\, x^{-2}\, dx$

$= \frac{3}{4}\, \frac{1}{-1}\, x^{-1} \Big|_1^5$

$= -\frac{3}{4}\, \frac{1}{x} \Big|_1^5$

$= \left(-\frac{3}{4}\, \frac{1}{(5)}\right) - \left(-\frac{3}{4}\, \frac{1}{(1)}\right)$

$= -\frac{3}{20} + \frac{3}{4} \times \frac{5}{5}$ common denominator

$= -\frac{3}{20} + \frac{15}{20}$

$= \frac{12}{20} = \boxed{\frac{3}{5}}$

3. $\int_0^1 e^{x/3}\,dx \longleftrightarrow$

$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$

$= \int_0^1 e^{\frac{1}{3}x}\, dx$

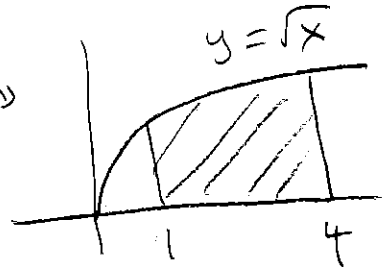
$= \frac{1}{1/3}\, e^{\frac{1}{3}x} \Big|_0^1$

$= 3e^{\frac{1}{3}x} \Big|_0^1$

$= \left(3e^{1/3(1)}\right) - \left(3e^{\frac{1}{3}(0)}\right)$

$= \boxed{3e^{1/3} - 3 \approx 1.1868}$

4. $\int_1^4 \sqrt{x}\, dx$



$$\int_1^4 x^{1/2}\, dx$$

$$= \frac{1}{3/2} x^{3/2} \Big|_1^4$$

$$= \frac{2}{3} x^{3/2} \Big|_1^4$$

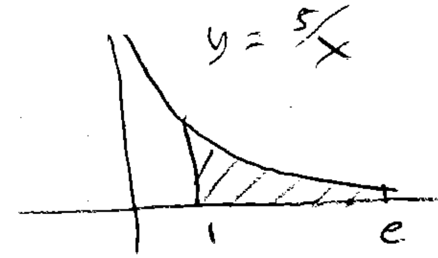
$$= \left(\frac{2}{3} (4)^{3/2}\right) - \left(\frac{2}{3} (1)^{3/2}\right)$$

$$= \frac{2}{3} \cdot 8 - 2/3$$

$$= \frac{16}{3} - 2/3$$

$$= \boxed{14/3}$$

5. $\int_1^e \frac{5}{x}\, dx$



$$\int_1^e 5 \cdot \frac{1}{x}\, dx$$

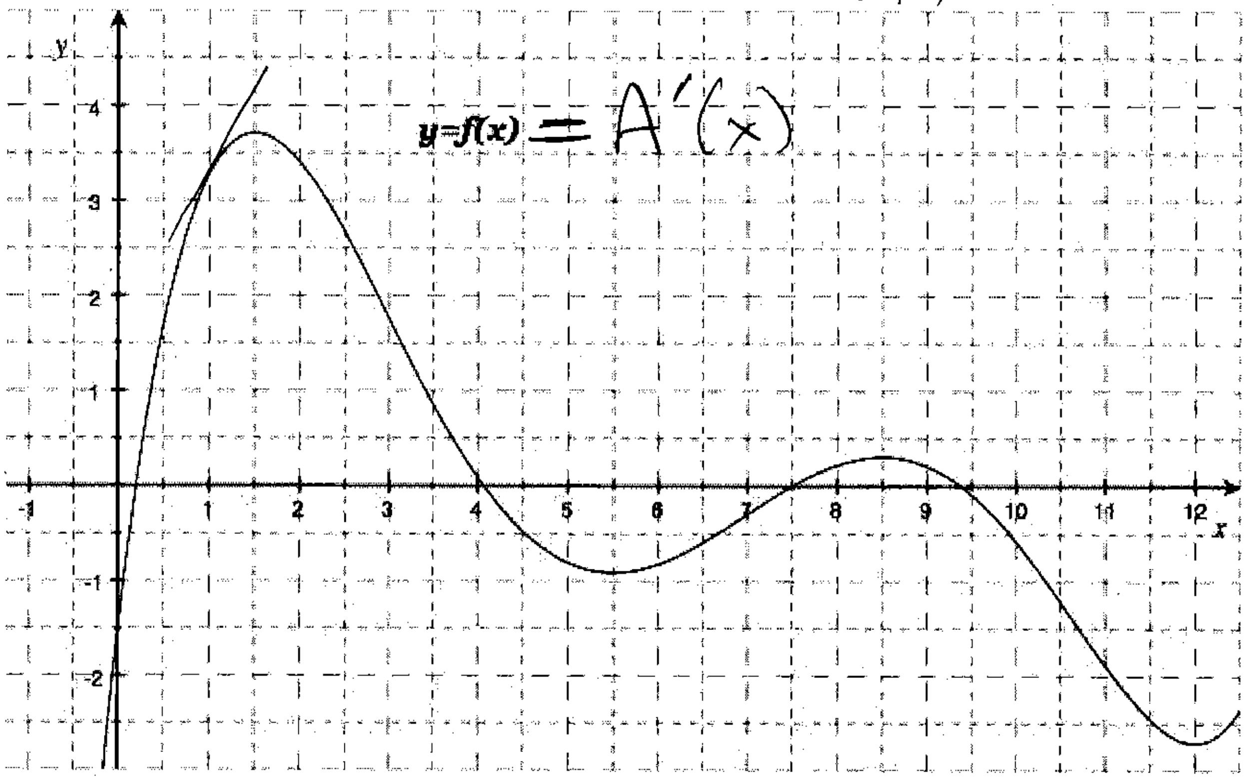
$$= 5 \ln(x) \Big|_1^e$$

$$= (5 \ln(e)) - (5 \ln(1))$$

$$= 5 - 0$$

$$= \boxed{5}$$

3. (18 points) Below is the graph of a function $y = f(x)$. $= A'(x)$



Define the function $A(m)$ by $A(m) = \int_0^m f(x)\,dx$.

NOTE: You do **not** need to show any work for the problems **on this page.**

A ↘ ↗ ↘ ↗ ↘

A′ − 0.25 + 4 − 7.5 + 9.3 −

(a) Name all values of m at which $A(m)$ has a local minimum.

ANSWER: $m =$ 0.25, 7.5

(b) Give the one-minute interval over which $A(m)$ increases the most. WHEN A′ IS BIGGEST

biggest slope of A

ANSWER: from 1 to 2

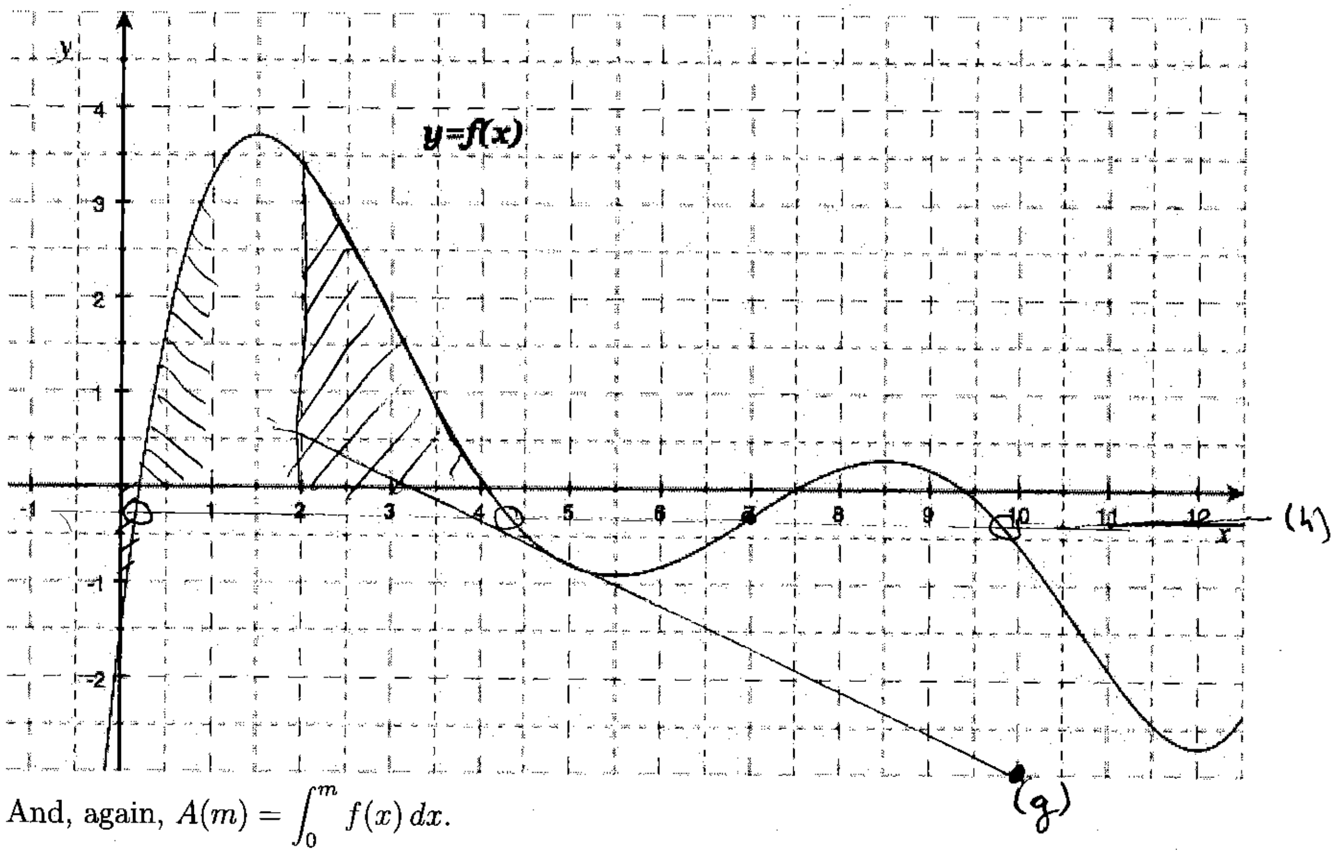
(c) True or False?

circle one

T	**F**	$A(2.51) > A(2.50)$	A′(2.50) IS POSITIVE ⇒ A INCREASING
T	**F**	$f(2.51) > f(2.50)$	f is decreasing at 2.5
T	**F**	$A(10.01) > A(10.00)$	A′(10) IS NEGATIVE ⇒ A DECREASING
T	**F**	$f'(1.00) > f'(1.01)$	slope of f is decreasing at 1.

(THIS PROBLEM IS CONTINUED ON THE NEXT PAGE.)

Here is the graph of $y = f(x)$ again.



And, again, $A(m) = \int_0^m f(x)\,dx$.

<u>NOTE:</u> The problems on this page **require some justification**: clearly mark points and lines on the graph, shade areas, show calculations of slopes and areas, etc.

(e) Compute $A(1)$.

AREA $\approx -\frac{1}{2}(1.5)(0.25) + \frac{1}{2}(0.75)(3.5) \approx 1.125$

ANSWER: $A(1) \approx$ 1.125

(f) Compute $A'(12)$. $= f(12) \approx -2.7$

ANSWER: $A'(12) \approx$ -2.7

(g) Compute $A''(5)$. $= f'(5) =$ slope at 5 $\quad \frac{-0.75 - -3}{5 - 10} =$

TWO POINTS ON TANGENT LINE $(10, -3)$ $(5, -0.75)$

ANSWER: $A''(5) \approx$ -0.45

(h) Name a value of x at which $f(x) = \underbrace{f(7)}_{\text{HEIGHT AT 7}}$.

ANSWER: $x =$ 0.2, or 4.25, or 9.75

(i) Compute $A(4) - A(2)$.

$A(4) - A(2) = \int_2^4 f(x)\,dx = \frac{1}{2}(2)(3.5)$

$=$ area from 2 to 4

ANSWER: $A(4) - A(2) =$ ≈ 3.5